MODEL CATEGORIES

Abstract.

1. Lecture 1

All categories are assumed to be small • with limits and colimits.

Given a category $\mathcal{C}$ we can construct the arrow category $Arr(\mathcal{C})$• whose objects are arrows in $\mathcal{C}$ and morphisms are commutative squares.

small or locally small?

Hovey uses Map($\mathcal{C}$)

Definition 1.1. Let $\mathcal{C}$ be a category.

(1) A morphism f in $\mathcal{C}$ is said to be a *retract* of a map g in $\mathcal{C}$ if there is a commutative diagram of the form,

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

(2) A *functorial factorisation* is an ordered pair (α, β) of functors $Arr(\mathcal{C}) \to Arr(\mathcal{C})$ such that $f = \beta(f) \circ \alpha(f)$ for all f in $Arr(\mathcal{C})$. That is, any $A \xrightarrow{f} B$ in $Arr(\mathcal{C})$ can be decomposed as $A \xrightarrow{\beta(f)} C \xrightarrow{\alpha(f)} B$, where C is some object in $\mathcal{C}$.

(3) Let $i : A \to B$ and $p : X \to Y$ be morphism in $\mathcal{C}$. We say that i has *left lifting property with respect to* p and that p has *right lifting property with respect to* i if for every commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 1.2. A *model structure* on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibration, and fibrations, and two functorial factorisations (α, β) and (γ, δ) satifying the following properties:

(1) (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g, and gf are weak equivalences, then so is the third.
(2) (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.
(3) (Lifting) Define a map to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations the left lifting property with respect to trivial fibrations.
(4) (Factorisation) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Essentially, (4) says that any morphism in $\mathcal{C}$ can be factorised as a cofibration followed by a trivial fibration, or a trivial cofibration followed by a fibration:

$$
\begin{aligned}
f &= \underset{\text{Trivial Fibration}}{\beta(f)} \circ \underset{\text{Cofibration}}{\alpha(f)} \\
&= \underset{\text{Fibration}}{\delta(f)} \circ \underset{\text{Trivial Cofibration}}{\gamma(f)}
\end{aligned}
$$

Definition 1.3. A category $\mathcal{C}$ with a model structure and in which all small limits and colimits exist is called a *model category*.

Some remarks about the dual model structure, product structure. Some examples (perhaps non-explicit).

Lemma 1.4. *(The Retract Argument). Let f be a morphism in $\mathcal{C}$ such that $f = p \circ i$, and f has the left lifting property with respect p. Then, f is a retract of i.*

Proof. Since f has the left lifting property with respect to p, we have the following commutative diagram,

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
{\scriptstyle f}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle p} \\
C & = & C
\end{array}
$$

Then, the following diagram finishes the proof,

$$
\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{r} & B & \xrightarrow{p} & C
\end{array}
$$

□

Lemma 1.5. *f is a cofibration (trivial cofibrations) if and only if f has the left lifting property with respect to trivial fibrations (fibrations).*

Proof. Clearly, cofibration have the left lifting property with respect to trivial fibrations. For the converse, let f be morphism which has the left lifting property with respect to trivial fibrations. Then, using the functorial factorisation, $f = \beta(f) \circ \alpha(f)$. Since, f has the left lifting property with respect to $\beta(f)$, by the retract argument f is a retract of $\alpha(f)$ and hence, a trivial fibration. □

Corollary 1.6. *Let $\mathcal{C}$ be a model category. Cofibration (trivial cofibrations) are closed under pushouts. Dually, fibrations (trivial fibrations) are closed under pullbacks.*

Proof. Follows from the universal property of pushouts (pullbacks). □

Remark 1.7. If $\mathcal{C}$ is a model category, it has both an initial object (the colimit of the empty diagram) and a final object (the limit of the empty diagram). An object of $\mathcal{C}$ is called *cofibrant* if the map from the inital object 0 to it is a cofibration. Dually, an object is called *fibrant* if the map to the final object from it is a fibration.

Moreover, if $0 \xrightarrow{f} B$ is any object in $\mathcal{C}$, we have a functorial factorisation $0 \xrightarrow{\alpha(f)} B' \xrightarrow{\beta(f)} B$, with $\alpha(f)$ a cofibration. Then, B' is called the *cofibrant replacement* of B. The notion of a fibrant replacement is defined similarly by considering the map to final object and the functorial factorisation (γ, δ).

Lemma 1.8. *(Ken Brown's lemma). Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with weak equivalences (that satisfy the two out of three property). Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which takes trivial cofibrations between cofibrant objects • to weak equivalences. Then, F takes all weak equivalences between cofibrant objects to weak equivalences.*

This has to be defined

Proof. Let $f : A \to B$ be a weak equivalence between cofibrant objects. We wish to show that $F(f)$ is also a weak equivalence.

Let $A \amalg B$ denote the pushout of A, B over the initial object. We have the map $(f, Id_B) : A \amalg B \to B$. Using the functorial factorisation, this can be factored as a cofibration q followed by a trivial fibration p. This gives us the following commutative diagram,•

This diagram will need positioning

Note that i_1, i_2, being pushouts of cofibrations, are cofibrations themselves. By the 2-out-of-3 property, as Id_B, p are weak equivalences, so is qi_2. Similarly, qi_1 is a weak equivalence as f, p are weak equivalences. Thus, qi_1 and qi_2 are trivial cofibrations. So, $F(qi_1)$ and $F(qi_2)$ are weak equivalences. As, $F(pqi_2) = F(Id_B)$, we see that $F(p)$ is a weak equivalence. Hence, $F(f) = F(pqi_1)$ is also a weak equivalence □

2. Lecture 2

Definition 2.1. Suppose $\mathcal{C}$ is a category with a subcategory of equivalences $\mathcal{W}$. We define the homotopy "category"• $\operatorname{Ho}\mathcal{C}$ as follows. Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of arrows in $\mathcal{W}$. An object of $F(\mathcal{C}, \mathcal{W}^{-1})$ is an object of $\mathcal{C}$, and a morphism is a finite string of composable arrows $(f_1, f_2, \ldots, f_n)$, where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of $\mathcal{W}$. The empty string at a particular object is the identity at that object, and composition is defined by concatenation of strings. Now, define $\operatorname{Ho}\mathcal{C}$ to be the quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations $1_A = (1_A)$ for all object A, $(f, g) = (g \circ f)$ for all composable arrows f, g of $\mathcal{C}$, and $1_{\mathrm{dom} w} = (w, w^{-1})$ and $1_{\mathrm{codom} w} = (w^{-1}, w)$ for all $w \in \mathcal{W}$.

pre-category?

Note that there is a functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ which is identity on objects and takes morphisms of $\mathcal{W}$ to isomorphisms. Some discussion of duals and products.

The category $\operatorname{Ho}\mathcal{C}$ has the following universal property.

Lemma 2.2. *Let $\mathcal{C}$ be a category with a subcategory $\mathcal{W}$.*

1. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathcal{W}$ to isomorphisms, then there is a unique functor $\operatorname{Ho} F : \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ such that $(\operatorname{Ho} F) \circ \gamma = F$.*
2. *Suppose $\delta : \mathcal{C} \to \mathcal{E}$ is a functor that takes maps of $\mathcal{W}$ to isomorphisms and enjoys the universal property of part(i). Then there is a unique isomorphism $F : \operatorname{Ho}\mathcal{C} \to \mathcal{E}$ such that $F \circ \gamma = \delta$.*
3. *The correspondence of part(i) induces an isomorphism of categories between the category of functors $\operatorname{Ho}\mathcal{C} \to \mathcal{D}$ and natural transformations and the category of functors $\mathcal{C} \to \mathcal{D}$ that take maps of $\mathcal{W}$ to isomorphisms and natural transformations.*

Lemma 2.3. *Suppose $\mathcal{C}$ is a model category. Consider the following three subcategories:*

$$\begin{pmatrix} \mathcal{C}_c & = & \text{cofibrant objects of } \mathcal{C}. \\ \mathcal{C}_f & = & \text{fibrant objects of } \mathcal{C} \\ \mathcal{C}_{cf} & = & \text{simultaneously cofibrant and fibrant objects of } \mathcal{C}. \end{pmatrix}$$

Then the inclusion functors equivalence of categories,

$$\begin{array}{ccccc} & & \operatorname{Ho}\mathcal{C}_c & & \\ & \nearrow & & \searrow & \\ \operatorname{Ho}\mathcal{C}_{cf} & & & & \operatorname{Ho}\mathcal{C} \\ & \searrow & & \nearrow & \\ & & \operatorname{Ho}\mathcal{C}_f & & \end{array}$$

Definition 2.4. Let $\mathcal{C}$ be a model category, and $f, g : B \to X$ be two maps.

(1) A *cylinder object* for B is a factorisation of the fold map $\nabla : B \amalg B \to B$ into a cofibration $B \amalg B \overset{i_0+i_1}{\to} B'$ followed by a weak equivalence $B' \overset{s}{\to} B$.
(2) A *path object* for X is a factorisation of the diagonal map $\Delta : X \to X \times X$ into a weak equivalence $X \overset{r}{\to} X'$ followed by a fibration $X' \overset{p_0,p_1}{\to} X \times X$.
(3) A *left homotopy* from f to g is a map $H : B' \to X$ for some cylinder object B' for B such that $Hi_0 = f$ and $Hi_1 = g$. We write $f \overset{l}{\sim} g$, if a left homotopy exists.
(4) A *right homotopy* from f to g is a map $K : B \to X'$ for some path object X' for X such that $p_0K = f$ and $p_1K = g$. We write $f \overset{r}{\sim} g$, if a right homotopy exists.
(5) We say that f and g are *homotopic*, and write $f \sim g$ is they are both left and right homotopic.
(6) f is a *homotopy equivalence* if there is a map $h : X \to B$ such that $hf \sim 1_B$ and $fh \sim 1_X$.

3. Lecture 3

Proposition 3.1. *Let $\mathcal{C}$ be a model categorym and $f, g : B \to X$ be two maps.*

(1) *If $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \to B$, then $fh \overset{r}{\sim} gh$.*
(2) *If X is fibrant, $f \overset{l}{\sim} g$, and $h : A \to B$, then $fh \overset{l}{\sim} gh$. Dually, if B is cofibrant, $f \overset{r}{\sim} g$, and $h : X \to Y$, then $hf \overset{r}{\sim} hg$.*
(3) *If B is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation on $\mathcal{C}(B, X)$.*
(4) *If B is cofibrant and $h : X \to Y$ is a trivial fibration or a weak equivalence of fibrant objects, then h induces an isomorphism,*

$$\mathcal{C}(B, X)/ \overset{l}{\sim} \overset{\cong}{\to} \mathcal{C}(B, Y)/ \overset{l}{\sim} .$$

Dually, if X is fibrant and $h : A \to B$ is a trivial cofibration or a weak equivalence of cofibrant objects, then h induces an isomorphism,

$$\mathcal{C}(B, X)/ \overset{r}{\sim} \overset{\cong}{\to} \mathcal{C}(B, Y)/ \overset{r}{\sim} .$$

(5) *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, then there is a right homotopy $K : B \to X'$ from f to g. Dually, if X is a fibrant object, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$, and there is a left homotopy from f to g using any cylinder object for B.*

Corollary 3.2. *If $\mathcal{C}$ is a model category, B is a cofibrant object, and X is a fibrant object, then left homotopy and right homotopy relations coincide on $\mathcal{C}(B, X)$ and are equivalence relations on it.*

Corollary 3.3. *The homotopy relation in $\mathcal{C}_{cf}$ is an equivalence relation. Hence the category $\mathcal{C}_{cf}/\sim$ exits.*

4. Lecture 4

Proposition 4.1. *A map in $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Theorem 4.2. *Let $\mathcal{C}$ be a model category. Let $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ denote the canonical functor, Q denote the cofibrant replacement functor of $\mathcal{C}$ and R denote the fibrant replacement functor.*

(1) *The inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \overset{\cong}{\to} \operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}.$$

(2) *There are natural isomorphisms*

$$\operatorname{Ho}\mathcal{C}(X, Y) \cong \mathcal{C}(QRX, QRY)/\sim \cong \mathcal{C}(QX, RY)/\sim .$$

(3) *if $f : A \to B$ is a map in $\mathcal{C}$ such that $\gamma(f)$ is an isomorphism, then f is a weak equivalence.*

Quillen Functors.

Definition 4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

(1) $F : \mathcal{C} \to \mathcal{D}$ is called a *left Quillen functor* if F is a left adjoint and preserves trivial cofibrations and cofibrations.
(2) $U : \mathcal{D} \to \mathcal{C}$ is called a *right Quillen functor* if U is a right adjoint and preserves trivial fibrations and fibrations.
(3) Let (F, U, ϕ) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. It is called a *Quillen adjunction* if F is a left Quillen functor.

Derived Functors.

Definition 4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

(1) Let $F : \mathcal{C} \to \mathcal{D}$ be a left Quillen functor. The *total left derived functor* LF of F is the composite
$$\operatorname{Ho}\mathcal{C} \stackrel{\operatorname{Ho}Q}{\to} \operatorname{Ho}\mathcal{C}_c \stackrel{\operatorname{Ho}F}{\to} \operatorname{Ho}\mathcal{D}.$$
(2) Similarly, the *the total right derived functor* RU of a right Quillen functor $U : \mathcal{D} \to \mathcal{C}$ is the composite
$$\operatorname{Ho}\mathcal{D} \stackrel{\operatorname{Ho}R}{\to} \operatorname{Ho}\mathcal{D}_f \stackrel{\operatorname{Ho}U}{\to} \operatorname{Ho}\mathcal{C}.$$

Quillen Equivalence.

Definition 4.5. A Quillen adjunction $(F, U, \phi) : \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\phi(f) : X \to UY$ is a weak equivalence in $\mathcal{C}$.

5. Lecture 5

6. Lecture 6

Definition 6.1. Let I be a class of maps in $\mathcal{C}$.

(1) A map in $\mathcal{C}$ is *I-injective* if it has the right lifting property with respect to all maps in I. The class of I-injective maps is denoted by I-inj.
(2) A map is *I-projective* if it has the left lifting property with respect to all maps in I. The class of I-projective maps is denoted by I-proj.
(3) A map is an *I-cofibration* if it the left lifting property with respect to every map in I-inj. The class of I-cofibrations is denoted by (I-inj)-proj.
(4) A map is an *I-fibration* if it has the right lifting property with respect to every map in I-proj. The class of I-fibrations is denoted by (I-proj)-inj.

Example 6.2. If $\mathcal{C}$ is a model category then, • align this list!
I = class of fibrations.
I-proj = class of trivial cofibrations.
(I-proj)-inj = class of fibrations.

Definition 6.3. Let $\mathcal{C}$ be a category with all small colimits and λ be an ordinal. A λ-sequence X is a colimit preserving functor $X : \lambda \to \mathcal{C}$,
$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots.$$
As X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map,
$$\operatorname{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$
is an isomorphism. The map $X_0 \to \operatorname{colim}_{\beta<\lambda}\mathrm{X}_\beta$ is called a *composition* of the λ-sequence.

If all $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$, lie in some collection $\mathcal{D}$ of morphisms of $\mathcal{C}$, then the composition $X_0 \to \operatorname{colim}_{\beta<\lambda}\mathrm{X}_\beta$ is called a *transfinite composition* of maps in $\mathcal{D}$.

Definition 6.4. Let I be a set of maps in $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is a composition of X and such that, for each β with $\beta + 1 < \lambda$, there is a pushout square,

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ \Big\downarrow{\scriptstyle g_\beta} & & \Big\downarrow \\ C_{\beta+1} & \longrightarrow & X_{\beta+1} \end{array}$$

with $g_\beta \in I$. The collection of relative I-cell complexes is denoted by I-cell. If $0 \to A$ is a relative I-cell complex then A is said to be an I-cell complex.

Definition 6.5. Let γ be a cardinal. An ordinal α is called *γ-filtered* if it is a limit ordinal and if for $A \subset \alpha$, $|A| \leq \gamma$ then $\sup A < \alpha$.

For a partially ordered set A, a subset B is called *cofinal* if for all $a \in A$ there is a $b \in B$ such that $a \leq b$. The *cofinality* of A is the least of the cardinalities of cofinal subsets of A. With this in mind we note that an ordinal α is γ-filtered if and only if the cofinality of α is greater than γ.

7. Lecture 7

Definition 7.1. Suppose $\mathcal{C}$ is a category with small colimits and $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$ and κ be an ordinal. We say that A is *κ-small* relative to $\mathcal{D}$ if for all κ-filtered ordinals λ and λ-sequences,

$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots.$$

such that $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map

$$\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda})$$

is an isomorphism.

Essentially, A is κ-small realtive to $\mathcal{D}$ if $\mathcal{C}(A, -)$ preserves colimits over κ-filtered ordinals.

Lemma 7.2. *Let I be a class of maps in a category $\mathcal{C}$ with all small colimits. Then I-cell $\subseteq$ I-cof.*

Lemma 7.3. *Suppose λ is an ordinal and $X : \lambda \to \mathcal{C}$ is a λ-sequence such that $X_\beta \to X_{\beta+1}$ is either a pushout of a map in I or an isomorphism. Then the transfinite composition of X is a relative I-cell complex.*

Lemma 7.4. *Let $\mathcal{C}$ be a category with all small colimits and I be a set of maps in $\mathcal{C}$. Then I-cell is closed under transfinite composition.*

Lemma 7.5. *Let $\mathcal{C}$ be a category with all small colimits and I be a set of maps in $\mathcal{C}$. Pushout of coproducts of maps in I is in I-cell.*

Theorem 7.6. (The Small Object Argument). *Let $\mathcal{C}$ contain all small colimits and I is a set of maps in $\mathcal{C}$. Suppose domains of maps of I are small relative to I-cell. Then there exists a functorial factorisation (γ, δ) on $\mathcal{C}$ such that for all $f \in \mathcal{C}$, $\gamma(f)$ is in I-cell and $\delta(f)$ is in I-inj.*

References

[1] Hovey, Mark. *Model Categories*, Mathematical Surveys and Monographs 63, AMS.